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MULTIMODAL OPTIMAL DESIGN OF A COMPRESSED COLUMN WITH RESPECT TO BUCKLING IN TWO PLANES

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Abstract—The aim of this paper is to present the optimal design of a compressed column with respect to buckling in two planes. The cross-section of the column is a rectangle and the ends of the column are elastically clamped. The optimization problem is to determine the two independent dimensions of the cross-section that minimize the total volume of the column under given external load and geometrical constraints. The necessity of the multi(uni-, bi-, tri-, quadri-)modal formulation of optimization is pointed out. The problem is solved using the Pontryagin maximum principle.

1. INTRODUCTION

The problem of optimal design of columns with respect to their stability has been discussed in many papers. Most of them concerned with the single eigenvalue-buckling load[1]. However, in many cases this unimodal formulation of the optimization problem is not sufficient. Kiusalaas[2] was one of the first to observe this phenomenon in the optimization problem of a column in an elastic medium. More detailed investigations have been performed by Olhoff and Rasmussen[3]. They pointed out the insufficiency of the unimodal formulation in the optimization problem of a clamped–clamped column compressed by an axial force. They arrived at the correct solution on the basis of a bimodal formulation of the problem, i.e. optimization with respect to a double eigenvalue connected with two fundamental forms of buckling. Similar problems are also dealt with in other papers, which have been discussed by Gajewski[4] and Masur[5]. In Ref. [5] the exact analytical solution of the Olhoff–Rasmussen problem[3] is presented. All the above-mentioned papers deal with optimal design with respect to buckling or vibration in one plane only.

The aim of this paper is to formulate a new problem of optimal design of a compressed column with respect to simultaneous buckling in two principal planes. The ends of the column are elastically clamped with different clamping flexibilities in each of the two perpendicular planes. However, it is assumed that in each plane clamping flexibilities at both ends are equal. Two forms of buckling in each plane, symmetric and antisymmetric connected with the lowest critical loads are considered.

The determination of two independent design variables, namely, the dimensions of a rectangular cross-section is an important problem which has not been discussed so far. Unimodal formulation is not possible in this case. It is indispensable to introduce the multi-(bi-, tri-, quadri-)modal formulation of the problem, i.e. optimization with respect to the critical load connected with two, three or four simultaneous buckling modes.

2. MATHEMATICAL DESCRIPTION

An elastic column of length l and of rectangular cross-section $B \times H$ shown in Fig. 1 is subjected to two equal concentrated forces of constant direction. The ends of the column are elastically clamped with, in general, different clamping flexibilities $\bar{\psi}_{xy}$, $\bar{\psi}_{xz}$ in two planes.

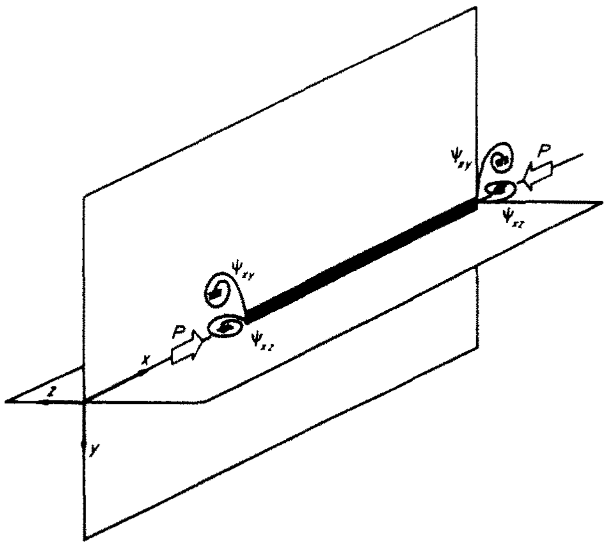

Fig. 1. The column with elastically clamped ends loaded by two concentrated forces.

The buckling state of the structure within the linear theory is described by the following differential equations

$$\frac{d^2}{dX^2}\left[EI_y(X)\frac{d^2W^{(k)}}{dX^2}\right]+P\frac{d^2W^{(k)}}{dX^2}=0 \tag{1}$$

$$\frac{d^2}{dX^2}\left[EI_z(X)\frac{d^2V^{(k)}}{dX^2}\right]+P\frac{d^2V^{(k)}}{dX^2}=0 \tag{2}$$

where E is Young's modulus, P the axial force, and I_y, I_z moments of inertia of the cross-section. Superscript k distinguishes suitable forms of buckling, symmetric ($k = 1$) and antisymmetric ($k = 2$). The functions W, V denote deflections of a column with respect to the coordinate system adopted here, and X is an independent variable. After introducing the dimensionless variables and parameters

$$x=\frac{X}{l},\quad v=\frac{V}{l},\quad w=\frac{W}{l},\quad s=\frac{I}{(1/12)A_0^2}$$
$$\beta=\frac{Pl^2}{(1/12)EA_0^2},\quad b=\frac{B}{\sqrt{A_0}},\quad h=\frac{H}{\sqrt{A_0}} \tag{3}$$

eqns (1) and (2) take the form

$$\frac{d^2}{dx^2}\left[s_y\frac{d^2w^{(k)}}{dx^2}\right]+\beta\frac{d^2w^{(k)}}{dx^2}=0 \tag{4}$$

$$\frac{d^2}{dx^2}\left[s_z\frac{d^2v^{(k)}}{dx^2}\right]+\beta\frac{d^2v^{(k)}}{dx^2}=0 \tag{5}$$

where a cross-sectional area A_0 is chosen so as to satisfy

$$V_{min}=A_0l,\qquad \int_0^1 b^*(x)h^*(x)\,dx=1 \tag{6}$$

V_{min} is a minimal volume of the column, and $b^*(x)$ and $h^*(x)$ optimal control functions.

Fourth-order differential equations (4) and (5) can be replaced by two sets of four first-order differential equations

$$\frac{d}{dx} w_i^{(k)} = \bar{D}_{ij} w_j^{(k)} \tag{7}$$

$$k = 1, 2, \qquad i,j = 1, 2, 3, 4$$

$$\frac{d}{dx} v_i^{(k)} = \bar{\bar{D}}_{ij} v_j^{(k)} \tag{8}$$

where

$$\bar{D}_{ij} = \begin{Bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\dfrac{1}{bh^3} & 0 \\ 0 & \beta & 0 & 1 \\ 0 & 0 & 0 & 0 \end{Bmatrix}, \qquad w_j^{(k)} = \begin{Bmatrix} w_1^{(k)} \\ w_2^{(k)} \\ w_3^{(k)} \\ w_4^{(k)} \end{Bmatrix} \tag{9}$$

$$\bar{\bar{D}}_{ij} = \begin{Bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\dfrac{1}{b^3 h} & 0 \\ 0 & \beta & 0 & 1 \\ 0 & 0 & 0 & 0 \end{Bmatrix}, \qquad v_j^{(k)} = \begin{Bmatrix} v_1^{(k)} \\ v_2^{(k)} \\ v_3^{(k)} \\ v_4^{(k)} \end{Bmatrix}. \tag{10}$$

Boundary conditions for the state equations (7) and (8) are

$$w_1^{(1)}(0) = w_3^{(1)}(0) - \frac{1}{\bar{\psi}_{xy}} w_2^{(1)}(0) = w_2^{(1)}\left(\frac{1}{2}\right) = w_4^{(1)}\left(\frac{1}{2}\right) = 0 \tag{11}$$

$$w_1^{(2)}(0) = w_3^{(2)}(0) - \frac{1}{\bar{\psi}_{xy}} w_2^{(2)}(0) = w_1^{(2)}\left(\frac{1}{2}\right) = w_3^{(2)}\left(\frac{1}{2}\right) = 0 \tag{12}$$

$$v_1^{(1)}(0) = v_3^{(1)}(0) - \frac{1}{\bar{\psi}_{xz}} v_2^{(1)}(0) = v_2^{(1)}\left(\frac{1}{2}\right) = v_4^{(1)}\left(\frac{1}{2}\right) = 0 \tag{13}$$

$$v_1^{(2)}(0) = v_3^{(2)}(0) - \frac{1}{\bar{\psi}_{xz}} v_2^{(2)}(0) = v_1^{(2)}\left(\frac{1}{2}\right) = v_3^{(2)}\left(\frac{1}{2}\right) = 0. \tag{14}$$

Boundary conditions (11)–(14) that distinguish symmetric and antisymmetric buckling modes are defined for $x = 0$ and $1/2$ due to the symmetry of the structure (in the prebuckling state). Such an approach is the most convenient to distinguish between the modes. Introducing a new variable y_0, eqn (6) may be written in the form

$$\frac{d}{dx} y_0 = bh \tag{15}$$

with conditions

$$y_0(0) = 0, \qquad y_0\left(\frac{1}{2}\right) = \frac{1}{2}.$$

3. FORMULATION OF THE OPTIMAL DESIGN PROBLEM

The problem of optimization is to determine two independent design variables $b^*(x)$ and $h^*(x)$ which satisfy the state equations with boundary conditions, normalization condition (15), geometrical constraints

$$b_1 \leqslant b^*(x) \leqslant b_2, \qquad h_1 \leqslant h^*(x) \leqslant h_2 \tag{16}$$

and minimize the total volume of the column under given external load

$$\int_0^1 bh\,\mathrm{d}x \to \min_{\beta=\mathrm{const.}}. \tag{17}$$

The necessary optimality condition is obtained by using the Pontryagin maximum principle. The justification of this approach for multimodal problems has been given for example by Teschner[6]. Introducing an adjoint state vector

$$\theta(\bar{\theta}_1^{(1)}\ldots\bar{\theta}_4^{(1)},\bar{\theta}_1^{(2)}\ldots\bar{\theta}_4^{(2)},\bar{\bar{\theta}}_1^{(1)}\ldots\bar{\bar{\theta}}_4^{(1)},\bar{\bar{\theta}}_1^{(2)}\ldots\bar{\bar{\theta}}_4^{(2)},\theta_0)$$

and assuming, in general, four simultaneous modes of buckling connected with the same buckling load, the Hamiltonian may be written in the form

$$H = \theta_0 bh + \bar{\theta}_i^{(k)}\bar{D}_{ij}w_j^{(k)} + \bar{\bar{\theta}}_i^{(k)}\bar{D}_{ij}v_j^{(k)},$$
$$k = 1,2, \qquad i,j = 1, 2, 3, 4 \tag{18}$$

where the summation convention holds. It is easy to prove that the problem under consideration is self-adjoint[4], hence eqn (18) takes the form

$$\begin{aligned} H = \theta_0 bh &+ \bar{C}^{(1)}\left[2w_4^{(1)}w_2^{(1)} + \frac{(w_3^{(1)})^2}{bh^3} + \beta(w_2^{(1)})\right] \\ &+ \bar{C}^{(2)}\left[2w_4^{(2)}w_2^{(2)} + \frac{(w_3^{(2)})^2}{bh^3} + \beta(w_2^{(2)})^2\right] \\ &+ \bar{\bar{C}}^{(1)}\left[2v_4^{(1)}v_2^{(1)} + \frac{(v_3^{(1)})^2}{b^3h} + \beta(v_2^{(1)})^2\right] \\ &+ \bar{\bar{C}}^{(2)}\left[2v_4^{(2)}v_2^{(2)} + \frac{(v_3^{(2)})^2}{b^3h} + \beta(v_2^{(2)})^2\right] \end{aligned} \tag{19}$$

where $\bar{C}^{(k)}$, $\bar{\bar{C}}^{(k)}$ are certain constants to be determined. In the case of two independent design variables b, h the necessary optimality condition has the form

$$\frac{\partial H}{\partial b} = 0, \qquad \frac{\partial H}{\partial h} = 0. \tag{20}$$

Solving conditions (20) we present the optimal functions $b^*(x)$, $h^*(x)$ in the following form

$$b^* = \left\{\lambda\frac{[\mu_2(v_3^{(1)})^2 + \mu_3(v_3^{(2)})^2]^2}{(w_3^{(1)})^2 + \mu_1(w_3^{(2)})^2}\right\}^{1/6} \tag{21}$$

$$h^* = \left\{\lambda\frac{[(w_3^{(1)})^2 + \mu_1(w_3^{(2)})^2]^2}{\mu_2(v_3^{(1)})^2 + \mu_3(v_3^{(2)})^2}\right\}^{1/6} \tag{22}$$

where

$$\lambda = \frac{4\bar{C}^{(1)}}{\theta_0}, \qquad \mu_1 = \frac{\bar{C}^{(2)}}{\bar{C}^{(1)}}, \qquad \mu_2 = \frac{\bar{\bar{C}}^{(1)}}{\bar{C}^{(1)}}, \qquad \mu_3 = \frac{\bar{\bar{C}}^{(2)}}{\bar{C}^{(1)}}. \tag{23}$$

In any case λ and μ_i have to be determined. Using eqns (7) and (8) we obtain

$$w_3^{(k)} = -bh^3\left[\frac{d^2}{dx^2}(w_1^{(k)})\right] \tag{24}$$

$$v_3^{(k)} = -b^3h\left[\frac{d^2}{dx^2}(v_1^{(k)})\right], \qquad k = 1,2 \tag{25}$$

and hence the optimality conditions may also be presented in the form

$$b^* = \left\{\alpha_1\left[\frac{d^2}{dx^2}(v_1^{(1)})\right]^2 + \alpha_2\left[\frac{d^2}{dx^2}(v_1^{(2)})\right]^2\right\}^{-1/2} \tag{26}$$

$$h^* = \left\{\alpha_3\left[\frac{d^2}{dx^2}(w_1^{(1)})\right]^2 + \alpha_4\left[\frac{d^2}{dx^2}(w_1^{(2)})\right]^2\right\}^{-1/2} \tag{27}$$

where new constants α_i ($i = 1,\ldots,4$) have to be determined. Expressions (26) and (27) can be obtained in a different way, namely using the calculus of variations. If some of μ_i, α_i vanish, we have a bi- or trimodal solution, whereas a unimodal solution is—in the present formulation—impossible.

4. METHOD OF SOLUTION

The solution to the problem formulated above (in Section 3) can be reduced to the solution of a non-linear boundary value problem, eqns (7), (8), and (11)–(15). At first, for given clamping flexibilities $\bar{\psi}_{xy}$, $\bar{\psi}_{xz}$ and geometrical constraints b_1, b_2, h_1, h_2 the bimodal optimal solution has been obtained by numerical integration of eqns (7), (8), (15) [$k = 1$] including (21), (22) and under the assumption that both buckling modes are connected with the same buckling load. Next, for the obtained optimal solution, the critical loads connected with the other buckling modes are calculated [$k = 2$]. If one of them has a value lower than "optimal", the bimodal solution is not correct. A trimodal formulation which equalizes three critical loads must be introduced. However, if in case of trimodal optimization the fourth critical load calculated for the trimodal solution has a lower value, then the quadrimodal formulation must be taken into consideration.

5. NUMERICAL EXAMPLES

As an illustration of the method presented in Section 4 for various clamping flexibilities $\bar{\psi}_{xy}$, $\bar{\psi}_{xz}$ and constraints $b_1 = h_1 = 0,1$, $b_2 = h_2 = 3,0$ optimal columns are obtained. Optimal $b^*(x)$, $h^*(x)$ functions and corresponding shapes for bimodal (Fig. 2) and trimodal formulations (Fig. 3) are presented. New parameters $\psi_{xy} = \bar{\psi}_{xy}/(1 + \bar{\psi}_{xy})$, $\psi_{xz} = \bar{\psi}_{xz}/(1 + \bar{\psi}_{xz})$ are introduced.

Figure 4 shows the range of application of suitable optimization modalities in terms of ψ_{xy}, ψ_{xz}. A characteristic feature of the problem is, that for $\psi_{xy} = \psi_{xz}$ and for square cross-sections, quadri-(bi-)modal and bi-(uni-)modal formulations lead to the same solutions, respectively. One can see that $\psi_{xy} = \psi_{xz} = 0,2$ is the critical value of the flexibility at which the unimodal solution switches to a bimodal one. The influence of clamping flexibilities on the bimodality was already investigated by Bochenek and Gajewski[7] in the optimization problem of an in-plane vibrating compressed bar.

A typical bimodal Olhoff–Rasmussen's solution for point (0,0) clamped–clamped column[3] is presented in Fig. 5 as a quadrimodal solution in the approach of this paper.

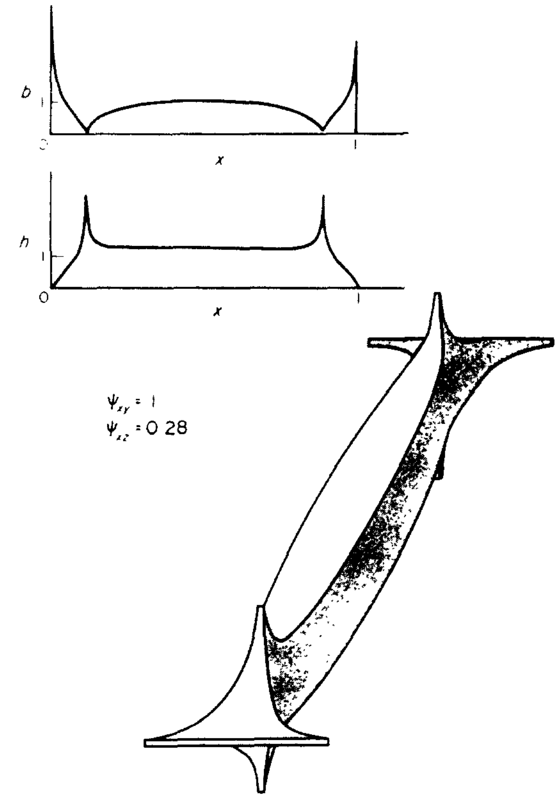

Fig. 2. Optimal column for $\psi_{xy} = 1.0$, $\psi_{xz} = 0.28$ and corresponding optimal functions $b^*(x)$, $h^*(x)$.

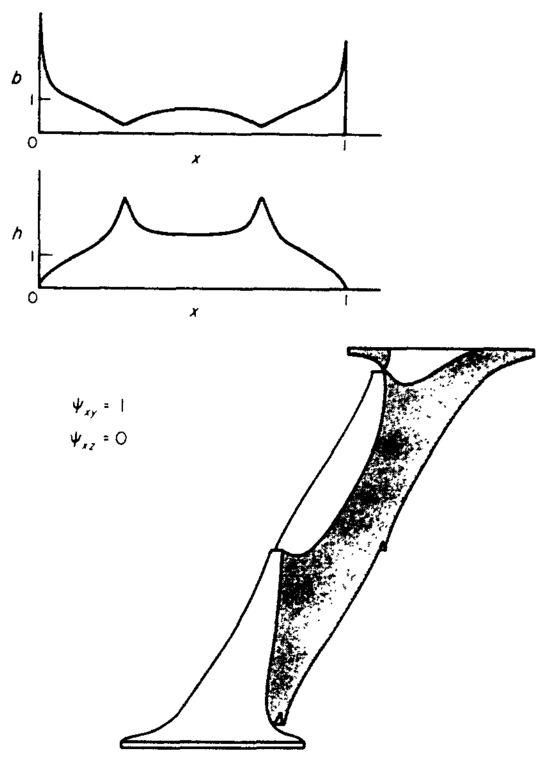

Fig. 3. Optimal column for $\psi_{xy} = 1.0$, $\psi_{xz} = 0$ and corresponding optimal functions $b^*(x)$, $h^*(x)$.

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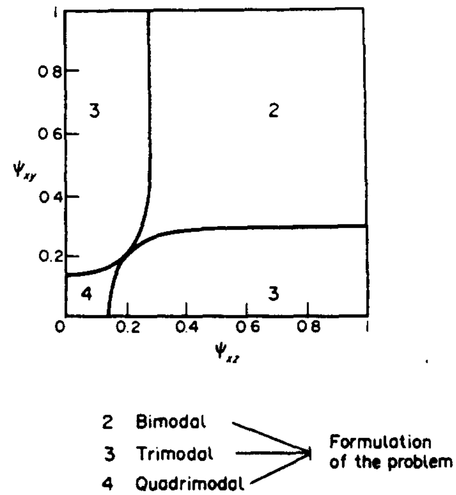

Fig. 4. The range of application of suitable optimization modalities in terms of ψ_{xy}, ψ_{xz}.

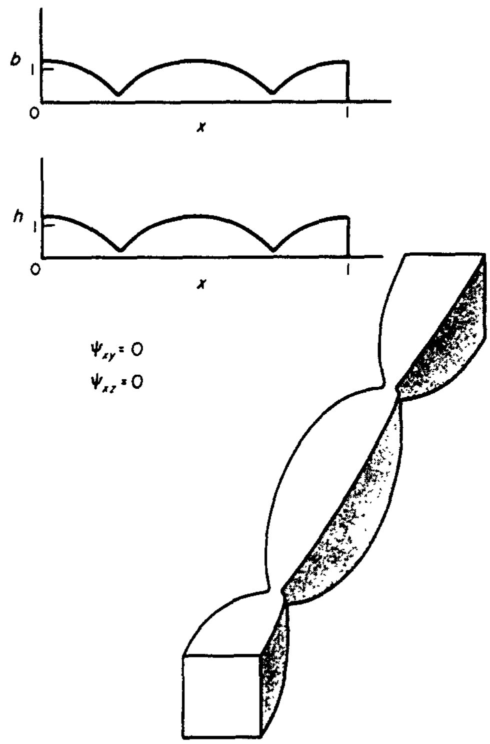

Fig. 5. Optimal column for $\psi_{xy} = 0$, $\psi_{xz} = 0$ and corresponding optimal functions $b^*(x)$, $h^*(x)$.

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